ATMOSPHERIC EFFECTS ON COHERENT LASER SYSTEMS

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Introduction and Summary

The final report documents the research work accomplished on the NASA Grant NSG 8037 entitled "Atmospheric Effects on Coherent Laser Systems" during the period June 1978 through May 1979. This report includes the material from the semi-annual report along with the additional analysis developed during the latter half of the Grant period. Most of the material from eq. (16) through eq. (33) is due to the research since December 1978.

The objective of the research is to investigate the effects of the truncation of the gaussian beam by the telescope on the signal-to-noise ratio of the NASA Laser Doppler System. The analysis presented in the next section deals with the finite beam effects neglecting the atmospheric effects. The well-known results for the case of infinite beam are derived from the finite beam formulation. The latter section develops the theory with the effects of the atmospheric turbulence and beam truncation. The results are expressed as finite integrals. These integrals require numerical integration. The integrals were programmed on the UNIVAC 1108. The program ran into unexpected computer memory problems and the results are fragmentary at this time.

Another objective was to gather data on the refractive index structure constant C_N using the CA-9 Space Averaging Anemometer. It took several months to obtain clearance from NASA to purchase the Anemometer and the associated equipment. Campbell Scientific, Inc. took almost 3 months to deliver the system. The system is in the process of being set-up over the roofs of Patton Hall and Carver Complex on the campus of Alabama A & M University.

A paper entitled "Laser Beam Propagation in Turbulent Atmosphere" is at the stage of galley proofs and will be published in the forthcoming issue of Proceedings C, Indian Academy of Sciences, Bangalore, India.

FINITE BEAM THEORY OF ATMOSPHERIC LDV SYSTEMS

Heterodyne detection of optical radiation scattered from aerosols has been the subject of several papers. We are concerned specifically with the detection of backscattered radiation and the extraction of velocity information of the target from the Doppler shift of the wave frequency. The NASA Laser Doppler System is of coaxial configuration with the same telescope acting as transmitting and receiving antenna. The intensity of laser radiation leaving the transmitting telescope has a Gaussian profile whose wings are cut off by the finite size of the telescope. It is simpler mathematically to treat the beam as an infinite Gaussian distributed field and ignore the finite size of the transmitter and receiver. Sonnenschein and Harrigan [1] analyzed the signal-to- noise relationships for a coaxial system that heterodynes the backscattered signal from atmospheric aerosols assuming infinite Gaussian fields. The real systems are of finite size and the radiation becomes a truncated Gaussian beam. The purpose of this work is to investigate the effect of the finite aperture size of the coaxial system on the signal-to-noise ratio and determine the conditions when the infinite wave analysis is valid.

There are a few research papers which deal with the effects of finite aperture size in practical systems. We refer to the papers relevant to the present work in the following. Classical heterodyne detection of an incoming optical signal by superposition of a beam from local oscillator at the finite receiver aperture is discussed by Fink [2] and Cohen [3] for plane or airy signal fields and by Takenaka, Tanaka and Fukumitsu [4] for Gaussian fields. Mandel and Wolf [5] considered the problem of detecting a coherent light beam in the form of a plane wave or Gaussian wave with an aperture of

circular or rectangular geometry. Halldorsson and Langerhole [6] considered the lidar set-up and calculated the form factors for atmospheric backscattering of laser radiation for misalignment of the axes and axial displacement of the aperture. Fried [7] considered the heterodyne detection of atmospherically distorted wavefront by a finite detector and showed that there is a limit to the useful collector diameter for an optical heterodyne system. Finally, Lutomirski and Buser [8] analyzed the mutual coherence function for a finite optical beam which contains the effects of source geometry and coherence loss in atmospheric propagation. The analysis of the coaxial system with finite aperture neglecting the atmospheric propagation effects is presented in the following.

The laser beam is transmitted along the Z axis and scattered back at the range point by the natural aerosols. The transmitter, receiver and the diffuse scatterers along with the coordinate system are shown in Fig. 1. The transmitter and receiver lens are in the xy plane and the scatterers are at a distance L.

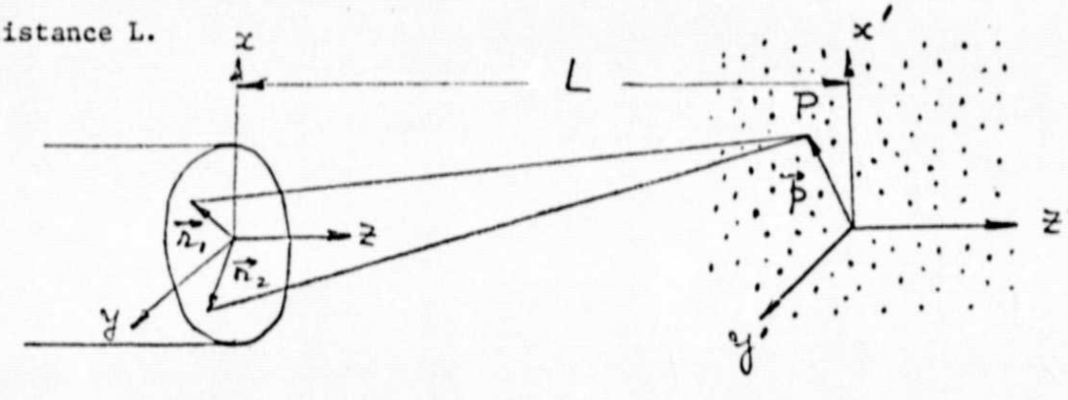

Fig. 1. Sketch of the telescope, scatterers and coordinate systems

We consider the coordinate vector $\vec{r}_1$ as a two-dimensional vector in the plane of the transmitting lens. The vector $\vec{r}_2$ is in the plane of the receiver lens. The vector $\vec{p}$ is a two-dimensional vector in the $x'y'$ scattering plane at the point P.

The wavefront leaving the transmitting lens is assumed to have a Gaussian amplitude distribution with standard deviation of a and focussed at range f from the transmitter. The transmitted wavefunction is given by

$$U(\vec{r}_1) = u_0\, e^{-\frac{r_1^2}{2a^2} - i\frac{k r_1^2}{2f}}, \qquad r_1 \le D/2$$
$$= 0 \qquad\qquad r_1 > D/2 \tag{1}$$

where $k = 2\pi/\lambda$ is the optical wave number, and $r_1 = |\vec{r}_1|$.

The wavefunction at the range point P before scattering takes place may be written using the Huygens-Fresnal principle:

$$U(\vec{p}) = -\frac{ik}{2\pi}\int_A U(\vec{r}_1)\, G(\vec{r}_1, \vec{p})\, d^2\vec{r}_1 \tag{2}$$

The term $G(\vec{p}, \vec{r}_1)$ is the field at the point P due to a point source at $\vec{r}_1$ and is given by

$$G(\vec{p}, \vec{r}_1) = \frac{e^{iks(\vec{p}, \vec{r}_1)}}{s(\vec{p}, \vec{r}_1)} \tag{3}$$

for a homogeneous medium where $s(\vec{p}, \vec{r}_1)$ is the geometric distance between the points $\vec{p}$ and $\vec{r}_1$. In the paraxial approximation, we may use

$$s(\vec{p}, \vec{r}_1) \simeq L\left[1 + \frac{(\vec{p}-\vec{r}_1)^2}{2L^2}\right] \tag{4}$$

and $$G(\vec{p}, \vec{r}_1) \simeq \frac{1}{L}\, e^{ikL + \frac{ik(\vec{p}-\vec{r}_1)^2}{2L}}$$

After substituting for G $(\vec{p}, \vec{r}_1)$ in eq. (2), we get

$$U(\vec{p}) = -\frac{iKe^{iKL}}{2\pi L}\int_A U(\vec{r}_1)\, e^{\frac{iK(\vec{p}-\vec{r}_1)^2}{2L}}\, d^2\vec{r}_1 \tag{5}$$

The integration in eq (5) is over the area of the receiver.

The laser beam will be back-scattered at point P. Applying the Huygens-Fresnel principle and paraxial approximation to the scattered beam, we obtain the wavefunction at the receiver given by

$$U(\vec{r}_2) = \left(\frac{\sigma'}{4\pi L^2}\right)^{1/2} U(\vec{p})\, e^{-\frac{iKr_2^2}{2f} + \frac{iK(\vec{p}-\vec{r}_2)^2}{2L} + i(KL + \Delta\omega t)} \tag{6}$$

where σ^1 is the radar crossection of a single aerosol.

The incident radiation is mixed with the local oscillator signal whose wavefunction is given by

$$U_{ref}(\vec{r}_2) = u_{ref}\, e^{-r_2^2/2a^2} \tag{7}$$

The signal current i_s is given by

$$i_s = 2\eta \int U(\vec{r}_2)\, U_{ref}(\vec{r}_2)\, W(\vec{r}_2)\, d^2\vec{r}_2 \tag{8}$$

where W $(\vec{r}_2)$ is the aperture function given by

$$\begin{aligned} W(\vec{r}_2) &= 1 \quad \text{for} \quad r_2 \leq D/2 \\ &= 0 \quad \text{for} \quad r_2 > D/2 \end{aligned}$$

We will now substitute equations (5), (6) and (7) in eq (8) and obtain the

following equation for the signal current:

$$i_s = -\frac{i\kappa\eta}{\pi L}\left(\frac{\sigma'}{4\pi L^2}\right)^{1/2} e^{i(2\kappa L+\Delta\omega t)} \int_A v(\vec{r}_1)\, e^{\frac{i\kappa}{2L}(\vec{\rho}-\vec{r}_1)^2} d^2\vec{r}_1 \,\cdot$$

$$\int V(\vec{r}_2)\, e^{\frac{i\kappa}{2L}(\vec{\rho}-\vec{r}_2)^2} W(\vec{r}_2)\, d^2\vec{r}_2 \tag{9}$$

and $$V(\vec{r}_2) = u_{ref}\, e^{-\frac{r_2^2}{2a^2} - \frac{i\kappa r_2^2}{2f}} \tag{10}$$

The signal power due to scattering from a single particle at P is proportional to the square of the current given by eq (10). To obtain the total power, we have to integrate over the probing volume. The pulsed system senses scattered radiation from a slab of air of thickness $\Delta L = c\tau/2$ in the paraxial approximation where c is the speed of light and τ is the pulse length.

Let I_s denote the total current due to scattering from all particles at a mean distance L. The total power is proportional to I_s^2.

$$I_s^2 = \int i_s^2\, dV = \frac{c\tau}{2}\int i_s^2\, d^2\vec{\rho} \tag{11}$$

In order to obtain the total signal power, we have to calculate i_s^2, which is written as a double integral. The ensemble average of the square of the signal current is given by:

$$\langle i_s^2\rangle = \frac{1}{2}\left(\frac{\kappa\eta}{\pi L}\right)^2 \frac{\sigma'}{4\pi L^2}\left[\int_A\int_A v(\vec{r}_1)v^*(\vec{r}_3)\, e^{\frac{i\kappa}{2L}\left\{(\vec{\rho}-\vec{r}_1)^2-(\vec{\rho}-\vec{r}_3)^2\right\}} d^2\vec{r}_1\, d^2\vec{r}_3\right]\cdot$$

$$\left[\int_A\int_A V(\vec{r}_2)V^*(\vec{r}_4)\, e^{\frac{i\kappa}{2L}\left\{(\vec{\rho}-\vec{r}_2)^2-(\vec{\rho}-\vec{r}_4)^2\right\}} W(\vec{r}_2)\, W(\vec{r}_4)\, d^2\vec{r}_2\, d^2\vec{r}_4\right] \tag{12}$$

The point $\vec{r}_3$ is in the plane of the transmitter and the point $\vec{r}_4$ is in the plane of the receiver. We will consider the integrals in each of the square brackets one after another. We introduce sum and difference coodinates and make use of the circular symmetry of the beam.

Define

$$\vec{R}_A \equiv \tfrac{1}{2}(\vec{r}_1 + \vec{r}_3) \qquad \vec{\rho}_A \equiv \vec{r}_1 - \vec{r}_3$$
$$\vec{r}_1 = \vec{R}_A + \vec{\rho}_A/2 \qquad \vec{r}_3 = \vec{R}_A - \vec{\rho}_A/2 \tag{13}$$

Substituting for $\vec{r}_1$ and $\vec{r}_3$, we obtain the following relations.

$$(\vec{p} - \vec{r}_1)^2 - (\vec{p} - \vec{r}_3)^2 = 2\,\vec{\rho}_A \cdot (\vec{R}_A - \vec{p}) \tag{13a}$$

$$\begin{aligned} v(\vec{r}_1)\,v^*(\vec{r}_3) &= v\left(\vec{R}_A + \frac{\vec{\rho}_A}{2}\right) v^*\left(\vec{R}_A - \frac{\vec{\rho}_A}{2}\right) \\ &= u_0^2\, e^{-\frac{R_A^2}{a^2} - \frac{\rho_A^2}{4a^2} - i\frac{k}{f}\vec{R}_A\cdot\vec{\rho}_A} \end{aligned} \tag{14}$$

Using the relations from eq (13) and (14), it is possible to write the following:

$$\int_A\int_A v(\vec{r}_1)\,v^*(\vec{r}_3)\, e^{\frac{ik}{2L}\left[(\vec{p}-\vec{r}_1)^2 - (\vec{p}-\vec{r}_3)^2\right]}\, d^2\vec{r}_1\, d^2\vec{r}_3$$

$$= u_0^2 \int_A e^{-\frac{\rho_A^2}{4a^2} - \frac{ik}{L}\vec{\rho}_A\cdot\vec{p}}\, d^2\vec{\rho}_A \cdot \int_A e^{-\frac{R_A^2}{a^2} + i\frac{k}{L}\left(1 - \frac{L}{f}\right)}\, d^2\vec{R}_A \tag{15}$$

$$= (2\pi u_o)^2 \int_0^{D} \rho_A \, d\rho_A \; e^{-\frac{\rho_A^2}{4a^2}} \; J_o\left(\frac{k}{L}\rho_A p\right) .$$

$$\int_0^{D/2} R_A \, dR_A \; e^{-\frac{R_A^2}{a^2}} \; J_o\left[\frac{k}{L}\left(1-\frac{L}{f}\right)\rho_A R_A\right] \qquad (15)$$

J_o is Bessel function of zero order.

Manipulations similar to those in eq (13), (14) and (15) yield the following results for the remaining integrals in eq (12).

Define sum and difference coordinates:

$$\vec{R}_B \equiv \frac{1}{2}(\vec{r}_2 + \vec{r}_4) \qquad \vec{\rho}_B \equiv \vec{r}_2 - \vec{r}_4 \qquad (16)$$

$$(\vec{p} - \vec{r}_2)^2 - (\vec{p} - \vec{r}_4)^2 = 2\,\vec{\rho}_B \cdot (\vec{R}_B - \vec{p}) \qquad (16a)$$

$$V(\vec{r}_2)\, V^*(\vec{r}_4) = u_{ref}^2 \; e^{-\frac{R_B^2}{a^2} - \frac{\rho_B^2}{4a^2} - \frac{ik}{f}\vec{\rho}_B \cdot \vec{R}_B} \qquad (17)$$

$$W(\vec{r}_2)\, W(\vec{r}_4) = W\left(\vec{R}_B + \frac{\vec{\rho}_B}{2}\right) W\left(\vec{R}_B - \frac{\vec{\rho}_B}{2}\right)$$

$$\int\int V(\vec{r}_2)\, V^*(\vec{r}_4)\; e^{\frac{ik}{2L}\left[(\vec{p} - \vec{r}_2)^2 - (\vec{p} - \vec{r}_4)^2\right]} \; W(\vec{r}_2)\, W(\vec{r}_4)\, d^2\vec{r}_2\, d^2\vec{r}_4$$

$$= u_{ref}^2 \int e^{-\frac{\rho_B^2}{4a^2} - \frac{ik}{L}\vec{\rho}_B \cdot \vec{p}} \; d^2\vec{\rho}_B \, . \qquad (18)$$

$$\int e^{-\frac{R_B^2}{a^2} + \frac{ik}{L}\vec{\rho}_B \cdot \vec{R}_B\left(1 - \frac{L}{f}\right)} \, W\left(\vec{R}_B + \frac{\vec{\rho}_B}{2}\right) W\left(\vec{R}_B - \frac{\vec{\rho}_B}{2}\right) d^2\vec{R}_B$$

The inner integral on the right hand side of eq (18) is the integration of the function

$$\exp\left[-\frac{R_B^2}{a^2}+\frac{ik}{L}\,\vec{\rho}_B\cdot\vec{R}_B\left(1-\frac{L}{f}\right)\right]$$

over the area of overlap of two circles, each of diameter D, with centers displaced at opposite ends of the vector $\vec{\rho}_B$.

Fig. 2 shows the area of overlap and the relevant vectors. O and O' are the

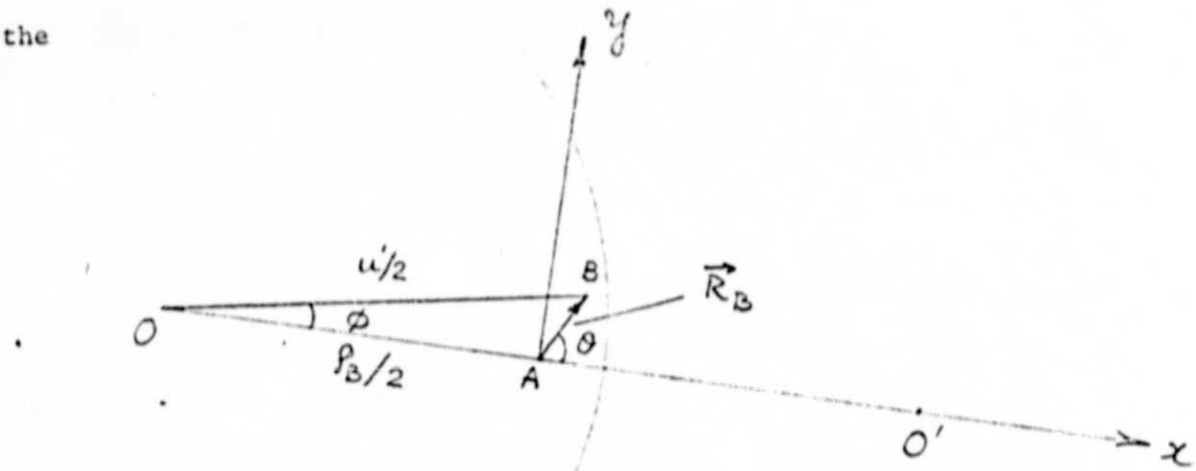

Fig. 2. Area of overlap

centers of the circles and the length OO' is equal to ρ_B. From the triangle OAB, we obtain the following.

$$\begin{aligned} R_B^2 &= \frac{1}{4}\left[(u')^2+\rho_B^2-2u'\rho_B\cos\phi\right] \\ &= \frac{D^2}{4}\left[u^2+x^2-2ux\cos\phi\right] \end{aligned} \tag{19}$$

where $u' = u'/D$ and $x = \rho_B/D$.

$$\vec{\rho}_{12} \cdot \vec{R}_B = \rho_B R_B \cos\theta = \rho_B\left(\frac{u'}{2}\cos\phi - \frac{\rho_A}{2}\right) \tag{20}$$

$$= \frac{D^2}{2}\, x\,(u\cos\phi - x)$$

$$\frac{K}{L}\left(1-\frac{L}{f}\right)\vec{\rho}_B \cdot \vec{R}_B = 2\alpha' x\,(u\cos\phi - x),\quad \alpha = \frac{KD^2}{4L},\ \alpha' = \alpha\left(1-\frac{L}{f}\right)$$

$$\text{Area element} = \frac{u'}{2}\,\frac{du'}{2}\,d\phi$$

$$= \frac{D^2}{4}\,u\,du\,d\phi \tag{21}$$

The inner integral may be written as follows using eq. (19), (20) and (21) over the hatched area in Fig. 2.

$$\int e^{-R_B^2/a^2 + i\frac{K}{L}\vec{\rho}_B\cdot\vec{R}_B\left(1-\frac{L}{f}\right)} W\left(R_B + \frac{\vec{\rho}_A}{2}\right) W\left(\vec{R}_B - \frac{\vec{\rho}_A}{2}\right) d^2\vec{R}_B$$

$$= \frac{D^2}{4}\int_0^{\cos^{-1}x} d\phi \int_{x/\cos\phi}^{1} e^{-\frac{D^2}{4a^2}\left(u^2 + x^2 - 2ux\cos\phi\right) + i\,2\alpha' x\,(u\cos\phi - x)}\, u\,du$$

Extending the integration over the entire area of overlap, we obtain the inner integral as $D^2\,\Gamma_2(x)\,e^{-\delta^2 x^2}$,

where

$$\Gamma_2(x) = \int_0^{\cos^{-1}x} d\phi \int_{x/\cos\phi}^{1} e^{-\delta^2\left(u^2 - 2ux\cos\phi\right)}\, u\,du \tag{22}$$

and $\delta = D/2a$

Using the results of eq (15) - (22) in eq (12), the square of the signal current becomes:

$$\langle i_s^2 \rangle = 16\,\sigma(\alpha\eta u_o u_{ref} D^2)^2 \int_0^1 e^{-y^2\delta^2} \Gamma_1(y) \cdot J_0(4\alpha y p') y\,dy \cdot$$

$$\cdot \int_0^1 e^{-2x^2\delta^2} \Gamma_2(x)\, J_0(4\alpha x p')\, x\, dx \tag{23}$$

and $y = \frac{\rho_A}{D}$, $p' = \frac{p}{D}$, $\Gamma_1(y) = \int_0^1 e^{-4\delta^2 z^2} J_0(4\alpha' y' z) z\,dz$

The integrals in eq (23) require numerical integration for finite aperture. However, they can be evaluated in closed form when the upper limits of integration are stretched to infinity which corresponds to an infinite beam and aperture. The infinite wave case is obtained by setting $D \to \infty$ in eq (15) and (18) and taking $W(\vec{r}) = 1$ for all values of $\vec{r}$. The result is

$$\langle i_s^2 \rangle = \frac{1}{2}(2\pi u_o)^2 (2\pi u_{ref})^4 \left(\frac{k\eta}{\pi L}\right)^2 \frac{\sigma'}{4\pi L^2} \cdot$$

$$\left[\int_0^\infty \rho\, d\rho\, e^{-\rho^2/4a^2} J_0\left(\frac{k}{L}\rho p\right) \cdot \int_0^\infty R\,dR\, e^{-R^2/a^2} J_0\left\{\frac{k}{L}\left(1-\frac{L}{f}\right)\rho R\right\}\right]^2 \tag{24}$$

The integrals are evaluated using the following relation:

$$\int_0^\infty x\,dx\, e^{-x^2/a^2} J_0(w x) = \frac{a^2}{2} e^{-\frac{w^2 a^2}{4}}$$

The result is the following:

$$\langle i_s^2 \rangle = 2\pi\sigma' \left(\frac{u_o u_{ref} k\eta}{L^2}\right)^2 \left(\frac{a^4}{1-\alpha'^2 a^4}\right)^2 e^{-\frac{2k^2 p^2 a^2}{L^2(1-\alpha'^2 a^4)}} \tag{25}$$

The aerosols are of varying sizes and we integrate eq (25) over the size distribution n(r) where r is the aerosol radius. The result is the replacement of σ' by σ which is given by

$$\sigma = \int_0^\infty \sigma'(r)\, n(r)\, dr$$

The radar crossection σ is related to the backscatter coefficient β by $\sigma = 4\pi\beta$. Substituting these relations, eq (25) becomes

$$\langle i_s^2 \rangle = 8\pi^2 \beta \left(\frac{u_o u_{ref} K \eta}{L^2} \right)^2 \left(\frac{a^4}{1+\alpha' a^4} \right)^2 e^{-\frac{2K^2 p^2 a^2}{L^2(1+\alpha'^2 a^4)}}.$$

From eq (11), the total current from all particles at a mean distance L for pulsed system is given by (dropping the angle brackets)

$$I_s^2 = \frac{c\tau}{2} \int_0^{2\pi} d\theta \int_0^{\infty} i_s^2 p\, dp = \pi c \tau \int_0^{\infty} i_s^2 p\, dp$$

$$= 8\pi^3 \beta c \tau \left(\frac{u_o u_{ref} k \eta}{L^2} \right)^2 \left(\frac{a^4}{1-\alpha'^2 a^4} \right)^2 \int_0^{\infty} p\, dp\; e^{-\frac{2K^2 p^2 a^2}{L^2(1+\alpha'^2 a^4)}} \tag{26}$$

$$= 2\pi^3 \beta c \tau \;\frac{u_o^2 u_{ref}^2 \eta^2}{L^2} \;\frac{a^6}{1+\alpha'^2 a^4}$$

Energy per pulse is defined by

$$E_T = \int_0^{\tau} P(t)\, dt = P_T \tau$$

where P_T is the average power. By the definition of the wavefunction, the number of photons N_T in the pulse is given by

$$N_T = \int_0^{\infty} v v^* d^2 \vec{r}_1 = \pi a^2 u_o^2 \tag{27}$$

Using the relation $P_T = h\nu N_T$, we obtain the relation

$$\pi a^2 u_o^2 = \frac{E_T}{h\nu\tau} \tag{28}$$

The noise power is proportional to the square of the noise current and is given by

$$i_n^2 = \frac{2\pi\eta}{\tau} a^2 u_{ref}^2 \tag{29}$$

We make use of eq (26), (27) and (28) to obtain:

$$\frac{S}{N} = \frac{\pi E_T \eta \beta c \tau}{h\nu} \; \frac{a^2}{L^2\left[1 + \frac{k^2 a^4}{L^2}\left(1 - \frac{L}{f}\right)^2\right]} \tag{30}$$

This is the same as that obtained in Ref. 1, if we note that $a^2 = R^2/2$ where R is the aperture radius.

For a finite beam, eq (27) and (20) should be modified as follows:

$$\begin{aligned} N_T = \int_A U U^* d^2\vec{r}_1 &= 2\pi u_0^2 \int_0^R e^{-\frac{r^2}{a^2}}\, r\, dr \\ &= \pi u_0^2 a^2 \left(1 - e^{-R^2/a^2}\right) \end{aligned} \tag{31}$$

$$i_n^2 = \frac{2\pi\eta}{\tau}\, u_{ref}^2\, a^2 \left(1 - e^{-R^2/a^2}\right) \tag{32}$$

Using eq (31) and (32) after a little algebra, the signal-to-noise ratio for finite beams is obtained as follows:

$$\begin{aligned} \frac{S}{N} = \frac{128\, \beta c \tau \eta \alpha^2 E_T \epsilon^4}{\left(1 - e^{-\epsilon^2}\right)^2 h\nu} \int_0^\infty p\, dp \int_0^1 y\, dy \,\cdot \\ \int_0^1 \Gamma_1(y)\, \Gamma_2(x)\, J_0(4\alpha y p')\, J_0(4\alpha x p') \exp\left[-\delta^2\left(y^2 + 2x^2\right)\right] x\, dx \end{aligned} \tag{33}$$

where $\epsilon = R/a$. ϵ represents the truncation of the Gaussian beam. For instance, if $\epsilon = 1$, the beam is truncated at the points where the intensity has dropped to e^{-1} As the value of ϵ increases, greater portions of the wings are included in the beam and if $\epsilon \to \infty$, there is no truncation.

PROPAGATION THROUGH ATMOSPHERIC TURBULENCE

The atmospheric turbulence introduces random fluctuations of amplitude and phase into the wavefunction. The turbulence effects are usually incorporated by modifying the term G ($\vec{p}$, $\vec{r}$) of eq (3) as follows:

$$G(\vec{p},\vec{r}) = \frac{\exp\left[ik\,s(\vec{p},\vec{r}) + \Psi(\vec{p},\vec{r})\right]}{s(\vec{p},\vec{r})}$$

$$\simeq \frac{1}{L}\exp\left[ikL + \frac{ik}{2L}(\vec{p}-\vec{r})^2 + \Psi(\vec{p},\vec{r})\right] \qquad (34)$$

where Ψ ($\vec{p}$, $\vec{r}$) describes the effects of the inhomogeneous medium on propagation of a spherical wave between the points $\vec{p}$ and $\vec{r}$ and is given by

$$\Psi(\vec{p},\vec{r}) = \chi(\vec{p},\vec{r}) + i\,S(\vec{p},\vec{r}) \qquad (35)$$

$\chi(\vec{p}, \vec{r})$ represents the perturbation in the log-amplitude of the wave and $S(\vec{p}, \vec{r})$ represents the perturbation of the phase between the points $\vec{p}$ and $\vec{r}$.

The Huygens-Fresnel formulation is modified using eq (34) and (35). Without further elaboration, we write the equation for the signal current using eq (8):

$$i_s = -\frac{ik\eta}{\pi L}\left(\frac{\sigma}{4\pi L^2}\right)^{1/2} e^{i(2kL+\Delta\omega t)} \cdot$$

$$\cdot \int_A U(\vec{r}_1)\, e^{\frac{ik}{2L}(\vec{p}-\vec{r}_1)^2 + \psi_1}\, d^2\vec{r}_1 \cdot \int V(\vec{r}_2)\, e^{\frac{ik}{2L}(\vec{p}-\vec{r}_2)^2 + \psi_2}\, W(\vec{r}_2)\, d^2\vec{r}_2 \qquad (36)$$

where $\Psi_1 = \Psi(\vec{p}, \vec{r}_1) = X_1 + i\,S_1$ represents the perturbations between the points $\vec{r}_1$ and $\vec{p}$, and $\psi_2 = \psi_2(\vec{p}, \vec{r}_2) = X_2 + i\,S_2$ represents the perturbations between the points $\vec{p}$ and $\vec{r}_2$.

The ensemble average of the square of the signal current is given by

$$\langle i_s^2 \rangle = \frac{\sigma}{4\pi L^2}\left(\frac{k\eta}{\pi L}\right)^2 \left[\iint_A v(\vec{r}_1)\, v^*(\vec{r}_3)\; e^{\frac{ik}{2L}\left[(\vec{\rho}-\vec{r}_1)^2-(\vec{\rho}-\vec{r}_3)^2\right]}\, d^2\vec{r}_1\, d^2\vec{r}_3 \cdot \right.$$

$$\iint_A v(\vec{r}_2)\, v(\vec{r}_4)\; e^{\frac{ik}{2L}\left[(\vec{\rho}-\vec{r}_2)^2-(\vec{\rho}-\vec{r}_4)^2\right]}\, W(\vec{r}_2)\, W(\vec{r}_4)\,\cdot$$

$$\left\langle e^{X_1+X_2+X_3+X_4+i(S_1+S_2+S_3+S_4)} \right\rangle d^2\vec{r}_2\, d^2\vec{r}_4 \tag{37}$$

The ensemble average in eq (37) may be evaluated using the following relations:

$$\left\langle e^{ax+by} \right\rangle = e^{\frac{1}{2}\left[a^2\langle (x-\bar{x})^2\rangle + b^2\langle (y-\bar{y})^2\rangle\right]} \cdot e^{a\bar{x}+b\bar{y}} \tag{38}$$

x and y are independent Gaussian-random variables with means of $\bar{x}$ and $\bar{y}$, and a and b are arbitrary constant.

Using eq (38), the following relation is obtained:

$$\left\langle e^{X_1+X_2+X_3+X_4+i(S_1+S_2+S_3+S_4)} \right\rangle = e^{2C_X(|\vec{r}_1-\vec{r}_2|)+2C_X(|\vec{r}_3-\vec{r}_4|)} \cdot e^{-\frac{1}{2}\left[D(|\vec{r}_1-\vec{r}_3|)+D(|\vec{r}_1-\vec{r}_4|)+D(|\vec{r}_2-\vec{r}_4|)+D(|\vec{r}_2-\vec{r}_3|)\right]} \cdot e^{+\frac{1}{2}\left[D(|\vec{r}_1-\vec{r}_2|)+D(|\vec{r}_3-\vec{r}_4|)\right]} \tag{39}$$

$C_X(r)$ is the log-amplitude covariance given by $C_X(r) = \langle X(\vec{r}_1)-\bar{X}\rangle\langle X(\vec{r}_2)-\bar{X}\rangle$

with $r = |\vec{r}_1 - \vec{r}_2|$.

$D(r)$ is the wave structure function and may be approximated by

$$D(r) = 2\left(\frac{r}{\rho_0}\right)^{5/3}, \quad \rho_0 = \left(0.546\, C_n^2 L K^2\right)^{-\frac{3}{5}}$$

The log-amplitude covariance for a spherical wave valid for short and long path lengths, and weak to strong turbulence is given by the following:

$$C_\chi(\rho_n, \sigma_t^2) = 2.95\, \sigma_t^2 \int_0^1 du \,\{u(1-u)\}^{5/6} \int_0^\infty dy \frac{\sin^2 y}{y^{11/6}} \cdot$$

$$\cdot \exp\{-\sigma_t^2 [u(1-u)]^{5/6} f(y)\}\, J_0\left[\left(\frac{4\pi y u}{1-u}\right)^{\frac{1}{2}} \rho_n\right]$$

$$\rho_n = \rho/\sqrt{\lambda L} \qquad \sigma_t^2 = 0.124\, K^{7/6} L^{11/6} C_N^2 \tag{40}$$

$$\text{and} \quad f(y) = 7.02\, y^{5/6} \int_{0.7y}^\infty d\xi\, \xi^{-\frac{8}{3}}[1 - J_0(\xi)]$$

In order to proceed further, a few approximations are made. Using the identity

$$(a-b+c-d)^p = (a-b)^p + (a-d)^p + (b-c)^p + (c-d)^p - (a-c)^p - (b-d)^p \quad \text{for } p = 1 \text{ or } 2$$

the structure functions are approximated as:

$$D(|\vec{r}_1 - \vec{r}_3|) + D(|\vec{r}_1 - \vec{r}_4|) + D(|\vec{r}_2 - \vec{r}_4|) + D(|\vec{r}_2 - \vec{r}_3|) - D(|\vec{r}_1 - \vec{r}_3|) - D(|\vec{r}_2 - \vec{r}_4|) \simeq D(|\vec{r}_1 - \vec{r}_3 + \vec{r}_2 - \vec{r}_4|) \tag{41}$$

The second approximation is to replace $C_X(|\vec{r}_1 - \vec{r}_2|)$ and $C_X(|\vec{r}_3 - \vec{r}_4|)$ by $C_x(0)$.

With these simplifications, eq (37) may be written as:

$$i_s^2 = \frac{\sigma}{4\pi L^2}\left(\frac{K\eta}{\pi L}\right)^2 e^{4C_\chi(0)} \left[\iint_A U(\vec{r}_1)\,U^*(\vec{r}_3)\, e^{\frac{iK}{2L}\{(\vec{\beta}-\vec{r}_1)^2-(\vec{\beta}-\vec{r}_3)^2\}}\, d^2\vec{r}_1\, d^2\vec{r}_3\,\cdot\right.$$

$$\iint_A V(\vec{r}_2)\,V(\vec{r}_4)\, e^{\frac{iK}{2L}\{(\vec{\beta}-\vec{r}_2)^2-(\vec{\beta}-\vec{r}_4)^2\}}\, W(\vec{r}_2)\,W(\vec{r}_4)\,\cdot$$

$$e^{-\left[\frac{|\vec{r}_1-\vec{r}_3+\vec{r}_2-\vec{r}_4|}{\rho_0}\right]^{\frac{5}{3}}}\, d^2\vec{r}_2\, d^2\vec{r}_4 \tag{42}$$

The sum and difference coordinates defined in eq (13) and (13a) will be utilized to simplify eq (42). The result is

$$\bar{i}_s^2 = 16\,\sigma(\alpha\eta u_0 u_{ref})^2 \int_0^D \rho_A\, d\rho_A\, e^{-\frac{\rho_A^2}{4a^2}}\, J_0\!\left(\frac{K}{L}p\rho\right)\cdot$$

$$\int_0^{D/2} R_A\, dR_A\, e^{-R^2/a^2}\, J_0\left[\frac{K}{L}\left(1-\frac{L}{f}\right)\rho_A R_A\right]\cdot$$

$$\int_0^D \rho_B\, d\rho_B\, e^{-\frac{\rho_B^2}{4a^2}}\, J_0\!\left(\frac{K}{L}\rho_B p\right)\Gamma_2(x)\, e^{-\left[\frac{|\vec{\rho}_A+\vec{\rho}_B|}{\rho_0}\right]^{5/3}} \tag{43}$$

Eq (43) requires numerical integration and may be programmed as a regular double integral.

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